

Mean-Field Backward Stochastic Differential Equations and Related Partial Differential Equations *

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Abstract

In [5] the authors obtained Mean-Field backward stochastic differential equations (BSDE) associated with a Mean-field stochastic differential equation (SDE) in a natural way as limit of some highly dimensional system of forward and backward SDEs, corresponding to a large number of “particles” (or “agents”). The objective of the present paper is to deepen the investigation of such Mean-Field BSDEs by studying them in a more general framework, with general driver, and to discuss comparison results for them. In a second step we are interested in partial differential equations (PDE) whose solutions can be stochastically interpreted in terms of Mean-Field BSDEs. For this we study a Mean-Field BSDE in a Markovian framework, associated with a Mean-Field forward equation. By combining classical BSDE methods, in particular that of “backward semi-groups” introduced by Peng [14], with specific arguments for Mean-Field BSDEs we prove that this Mean-Field BSDE describes the viscosity solution of a nonlocal PDE. The uniqueness of this viscosity solution is obtained for the space of continuous functions with polynomial growth. With the help of an example it is shown that for the nonlocal PDEs associated to Mean-Field BSDEs one cannot expect to have uniqueness in a larger space of continuous functions.

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1 Introduction

Classical mathematical mean-field approaches can be met in different fields, so in Statistical Mechanics and Physics (for instance, the derivation of Boltzmann or Vlasov equations in the kinetic gas theory) and in Quantum Mechanics and Quantum Chemistry (for instance, the density functional models or also Hartree and Hartree-Fock type models). In a recent series of papers (see [10] and their papers cited therein) Lasry and Lions extended the field of such mean-field approaches to problems in Economics and Finance, and also to the theory of stochastic differential games. Recently Buckdahn, Li and Peng [5] studied a special mean field problem in a purely stochastic approach and deduced a new kind of BSDEs which they called Mean-Field BSDEs. In order to be more precise, we consider the stochastic dynamics $X = (X_t)_{t \in [0, T]}$ of a particle

$$\begin{aligned} dX_t &= \sigma(t, X_t, X_t^1, \dots, X_t^N) dB_t + b(t, X_t, X_t^1, \dots, X_t^N) dt, \quad t \in [0, T], \\ X_0 &= x \in R^n, \end{aligned} \quad (1.1)$$

which is governed by a d -dimensional Brownian motion $B = (B_t)_{t \in [0, T]}$ and influenced by the positions X^1, \dots, X^N of N other particles that are supposed to be mutually independent of each other, also independent of the driving Brownian motion B , and of the same law as X . The time $T > 0$ is an arbitrarily fixed horizon. In [5] it is shown that, under appropriate assumptions, SDE (1.1) has a unique solution.

For the study of the limit behavior $N \rightarrow +\infty$ of the above SDE of rank N , the authors of [5] assumed that the coefficients $\sigma = \sigma^N$ and $b = b^N$ of SDE (1.1) of rank N have the form

$$\sigma^N(t, x, x^1, \dots, x^N) = \frac{1}{N} \sum_{i=1}^N \sigma(t, x^i, x), \quad b^N(t, x, x^1, \dots, x^N) = \frac{1}{N} \sum_{i=1}^N b(t, x^i, x),$$

$(t, x, (x^1, \dots, x^N)) \in [0, T] \times R^n \times R^{n \times N}$. The functions $\sigma : [0, T] \times R^n \times R^n \rightarrow R^{n \times d}$, $b : [0, T] \times R^n \times R^n \rightarrow R^n$ are supposed to be bounded, measurable and, moreover, Lipschitz in the space variables, uniformly with respect to the time variable. Denoting by $(X^{(N)}, X^{(N),1}, \dots, X^{(N),N}) := (X, X^1, \dots, X^N)$ the solution of SDE (1.1) with the coefficients σ^N and b^N , it is shown in [5] that the processes $X^{(N)}$ converge uniformly on the time interval $[0, T]$, in L^2 , to the unique continuous \mathbb{F}^B -adapted solution \hat{X} of the limit equation (Mean-Field SDE)

$$d\hat{X}_t = E \left[\sigma(t, \hat{X}_t, \mu) \right]_{\mu=\hat{X}_t} dB_t + E \left[b(t, \hat{X}_t, \mu) \right]_{\mu=\hat{X}_t} dt, \quad t \in [0, T], \quad \hat{X}_0 = x \quad (1.2)$$

(\mathbb{F}^B is the filtration generated by B).

With the SDE of rank N the authors of [5] associated a BSDE of the form

$$\begin{aligned} dY_t &= -f^N(t, X_t, X_t^1, \dots, X_t^N, Y_t, Y_t^1, \dots, Y_t^N, Z_t, Z_t^1, \dots, Z_t^N) dt + Z_t dB_t, \quad t \in [0, T], \\ Y_T &= \Phi^N(X_T, X_T^1, \dots, X_T^N), \end{aligned} \quad (1.3)$$

in which, in the same spirit as for the forward SDE of rank N , (X^i, Y^i, Z^i) , $1 \leq i \leq N$, is a sample of N mutually independent triplets which are independent of the driving Brownian motion B and obey the same law as the solution (X, Y, Z) ; recall that (X, X^1, \dots, X^N) is the solution of the above SDE (1.1) with coefficients σ^N and b^N . The coefficients f^N, Φ^N for the above BSDE are introduced in the same spirit as σ^N, b^N :

$$f^N(t, x, x^1, \dots, x^N, y, y^1, \dots, y^N, z, z^1, \dots, z^N) = \frac{1}{N} \sum_{i=1}^N f(t, x^i, x, y^i, y, z^i, z),$$

$$\Phi^N(x, x^1, \dots, x^N) = \frac{1}{N} \sum_{i=1}^N \Phi(x^i, x),$$

$t \in [0, T]$, $x, x^j \in R^n, y, y^j \in R, z, z^j \in R^d$, $1 \leq j \leq N$, where $f : [0, T] \times R^n \times R^n \times R \times R \times R^d \times R^d \rightarrow R$ and $\Phi : R^n \times R^n \rightarrow R$ are bounded, measurable and Lipschitz in the variables (x', x, y', y, z', z) , uniformly with respect to $t \in [0, T]$. In [5] it is proved that, with

$$(Y^{(N)}, Y^{(N),1}, \dots, Y^{(N),N}, Z^{(N)}, Z^{(N),1}, \dots, Z^{(N),N}) := (Y, Y^1, \dots, Y^N, Z, Z^1, \dots, Z^N)$$

denoting the solution of the above BSDE (1.3), the couple of processes $(Y^{(N)}, Z^{(N)})$, interpreted as random variable with values in $C([0, T]; R) \times L^2([0, T]; R^d)$, converges to the unique square integrable, \mathbb{F}^B -progressively measurable solution of the limit equation (called Mean-Field BSDE)

$$\begin{aligned} d\hat{Y}_t &= -E \left[f(t, \hat{X}_t, \mu, \hat{Y}_t, \lambda, \hat{Z}_t, \zeta) \right]_{\mu=\hat{X}_t, \lambda=\hat{Y}_t, \zeta=\hat{Z}_t} dt + \hat{Z}_t dB_t, \quad t \in [0, T], \\ Y_T &= E \left[\Phi(\hat{X}_T, \mu) \right]_{\mu=\hat{X}_T}, \end{aligned} \quad (1.4)$$

where \hat{X} is the solution of Mean-Field SDE (1.2).

The objective of the present paper is to deepen the investigation of the above Mean-Field BSDE. In a first leg we study the existence and uniqueness for Mean-Field BSDEs in a rather general setting, with drivers which, in difference to [5], are not necessarily deterministic coefficients. In addition to the existence and the uniqueness also the comparison principle for this new type of BSDEs is discussed and illustrated by examples.

The main objective of the paper concerns the study of Mean-Field problems in a Markovian setting. To be more precise, we investigate Mean-Field BSDEs associated with Mean-Field forward SDEs and partial differential equations (PDEs) whose solutions are described by them. The system dynamics we investigate is given by the following SDE

$$\begin{cases} dX_s^{t,\zeta} &= E[b(s, X_s^{0,x_0}, \mu)]_{\mu=X_s^{t,\zeta}} ds + E[\sigma(s, X_s^{0,x_0}, \mu)]_{\mu=X_s^{t,\zeta}} dB_s, \quad s \in [t, T], \\ X_t^{t,\zeta} &= \zeta. \end{cases} \quad (1.5)$$

Precise assumptions on the coefficients $b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ are given in the following sections.

With the above SDE we associate the BSDE:

$$\begin{cases} -dY_s^{t,\zeta} &= E[f(s, X_s^{0,x_0}, \mu, Y_s^{0,x_0}, \lambda, \zeta)]_{\mu=X_s^{t,\zeta}, \lambda=Y_s^{t,\zeta}, \zeta=Z_s^{t,\zeta}} ds - Z_s^{t,\zeta} dB_s, \quad s \in [t, T], \\ Y_T^{t,\zeta} &= E[\Phi(X_T^{0,x_0}, \mu)]_{\mu=X_T^{t,\zeta}}. \end{cases} \quad (1.6)$$

Under the assumptions on f and Φ that are introduced in Section 5, the above BSDE has a unique solution $(Y_s^{t,x}, Z_s^{t,x})_{s \in [t, T]}$ and we can define the deterministic function

$$u(t, x) = Y_t^{t,x}. \quad (1.7)$$

We prove that $u(t, x)$ is the unique viscosity solution in $C_p([0, T] \times \mathbb{R}^n)$ to the following nonlocal PDE

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) + Au(t, x) + E[f(t, X_t^{0, x_0}, x, u(t, X_t^{0, x_0}), u(t, x), Du(t, x) \cdot E[\sigma(t, X_t^{0, x_0}, x)])] = 0, \\ (t, x) \in [0, T] \times \mathbb{R}^n, \\ u(T, x) = E[\Phi(X_T^{0, x_0}, x)], \quad x \in \mathbb{R}^n, \end{cases} \quad (1.8)$$

with

$$Au(t, x) := \frac{1}{2} \text{tr}(E[\sigma(t, X_t^{0, x_0}, x)] E[\sigma(t, X_t^{0, x_0}, x)]^T D^2 u(t, x)) + Du(t, x) \cdot E[b(t, X_t^{0, x_0}, x)].$$

In particular, it is shown that the space $C_p([0, T] \times \mathbb{R}^n)$ is the optimal space in which the uniqueness can be got.

Our paper is organized as follows. Section 2 recalls some elements of the theory of BSDEs which are needed in what follows. Section 3 investigates the properties of general Mean-Field BSDEs. We first prove the uniqueness and existence of the solution of Mean-Field BSDE (Theorem 3.1) but also the comparison theorem (Theorem 3.2) and the converse comparison theorem (Theorem 3.3). Similarly we investigate Mean-Field Forward SDEs in Section 4. In Section 5 we investigate decoupled Mean-Field Forward-Backward SDEs (FBSDEs). Their value function u (see (5.4)) turns out to be a deterministic function which is Lipschitz in x (see (5.5)) and $\frac{1}{2}$ -Hölder continuous in t (Theorem 5.2). Moreover, it satisfies the dynamic programming principle (DPP) (see 5.10). A key element in the proof of the DPP is the use of Peng's backward semigroups (see [14]). We change slightly its definition; this allows to shorten the argument for the proof that u is a viscosity solution of the associated PDE (Theorem 6.1). Finally, the uniqueness of the viscosity solution in the space of continuous functions with polynomial growth is proved in Section 7.

2 Preliminaries

Let $\{B_t\}_{t \geq 0}$ be a d -dimensional standard Brownian motion defined over some complete probability space (Ω, \mathcal{F}, P) . By $\mathbb{F} = \{\mathcal{F}_s, 0 \leq s \leq T\}$ we denote the natural filtration generated by $\{B_s\}_{0 \leq s \leq T}$ and augmented by all P -null sets, i.e.,

$$\mathcal{F}_s = \sigma\{B_r, r \leq s\} \vee \mathcal{N}_P, \quad s \in [0, T],$$

where \mathcal{N}_P is the set of all P -null subsets and $T > 0$ a fixed real time horizon. For any $n \geq 1$, $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^n$. We also shall introduce the following both spaces of processes which are used frequently in what follows:

$\mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}) := \{(\psi_t)_{0 \leq t \leq T} \text{ real-valued } \mathbb{F}\text{-adapted c\`adl\`ag process} :$

$$E[\sup_{0 \leq t \leq T} |\psi_t|^2] < +\infty\};$$

$\mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^n) := \{(\psi_t)_{0 \leq t \leq T} \mathbb{R}^n\text{-valued } \mathbb{F}\text{-progressively measurable process} :$

$$\|\psi\|_2^2 = E[\int_0^T |\psi_t|^2 dt] < +\infty\}.$$

Let us now consider a function $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ with the property that $(g(t, y, z))_{t \in [0, T]}$ is \mathbb{F} -progressively measurable for each (y, z) in $\mathbb{R} \times \mathbb{R}^d$, and which is assumed to satisfy the following standard assumptions throughout the paper:

- (A1) There exists a constant $C \geq 0$ such that, dtdP-a.e., for all $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$,
 $|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|);$
(A2) $g(\cdot, 0, 0) \in \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R})$.

The following result on BSDEs is by now well known, for its proof the reader is referred to Pardoux and Peng [12].

Lemma 2.1. *Under the assumptions (A1) and (A2), for any random variable $\xi \in L^2(\mathcal{O}, \mathcal{F}_T, P)$, the BSDE*

$$y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T z_s dB_s, \quad 0 \leq t \leq T, \quad (2.1)$$

has a unique adapted solution

$$(y_t, z_t)_{t \in [0, T]} \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^d).$$

In what follows, we always assume that the driving coefficient g of a BSDE satisfies (A1) and (A2).

Let us remark that Lemma 2.1 remains true when assumption (A1) is replaced by weaker assumptions, for instance those studied in Bahlali [1], Bahlali, Essaky, Hassani and Pardoux [2] or Pardoux and Peng [13]. However, here, for the sake of simplicity of the calculation we prefer to work with the Lipschitz assumption.

We also shall recall the following both basic results on BSDEs. We begin with the well-known comparison theorem (see Theorem 2.2 in El Karoui, Peng and Quenez [7]).

Lemma 2.2. *(Comparison Theorem) Given two coefficients g_1 and g_2 satisfying (A1) and (A2) and two terminal values $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P)$, we denote by (y^1, z^1) and (y^2, z^2) the solution of the BSDE with data (ξ_1, g_1) and (ξ_2, g_2) , respectively. Then we have:*

- (i) *(Monotonicity) If $\xi_1 \geq \xi_2$ and $g_1 \geq g_2$, a.s., then $y_t^1 \geq y_t^2$, a.s., for all $t \in [0, T]$.*
- (ii) *(Strict Monotonicity) If, in addition to (i), we also assume that $P(\xi_1 > \xi_2) > 0$, then $P\{y_t^1 > y_t^2\} > 0$, $0 \leq t \leq T$, and in particular, $y_0^1 > y_0^2$.*

Using the notation introduced in Lemma 2.2 we now suppose that, for some $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying (A1) and (A2), the drivers g_i , $i = 1, 2$, are of the form

$$g_i(s, y_s^i, z_s^i) = g(s, y_s^i, z_s^i) + \varphi_i(s), \quad \text{dsdP-a.e.,}$$

where $\varphi_i \in \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R})$. Then, for terminal values ξ_1, ξ_2 belonging to $L^2(\Omega, \mathcal{F}_T, P)$ we have the following

Lemma 2.3. *The difference of the solutions (y^1, z^1) and (y^2, z^2) of the BSDE with data (ξ_1, g_1) and (ξ_2, g_2) , respectively, satisfies the following estimate:*

$$\begin{aligned} & |y_t^1 - y_t^2|^2 + \frac{1}{2} E[\int_t^T e^{\beta(s-t)} (|y_s^1 - y_s^2|^2 + |z_s^1 - z_s^2|^2) ds | \mathcal{F}_t] \\ & \leq E[e^{\beta(T-t)} |\xi_1 - \xi_2|^2 | \mathcal{F}_t] + E[\int_t^T e^{\beta(s-t)} |\varphi_1(s) - \varphi_2(s)|^2 ds | \mathcal{F}_t], \quad P\text{-a.s., for all } 0 \leq t \leq T, \end{aligned}$$

where $\beta = 16(1 + C^2)$.

For the proof the reader is referred to Proposition 2.1 in El Karoui, Peng and Quenez [7] or Theorem 2.3 in Peng [14].

3 Mean-Field Backward Stochastic Differential Equations

This section is devoted to the study of a new type of BSDEs, the so called Mean-Field BSDEs.

Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P)$ be the (non-completed) product of (Ω, \mathcal{F}, P) with itself. We endow this product space with the filtration $\bar{\mathbb{F}} = \{\bar{\mathcal{F}}_t = \mathcal{F} \otimes \mathcal{F}_t, 0 \leq t \leq T\}$. A random variable $\xi \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ originally defined on Ω is extended canonically to $\bar{\Omega}$: $\xi'(\omega', \omega) = \xi(\omega')$, $(\omega', \omega) \in \bar{\Omega} = \Omega \times \Omega$. For any $\theta \in L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ the variable $\theta(., \omega) : \Omega \rightarrow \mathbb{R}$ belongs to $L^1(\Omega, \mathcal{F}, P)$, $P(d\omega)$ -a.s.; we denote its expectation by

$$E'[\theta(., \omega)] = \int_{\Omega} \theta(\omega', \omega) P(d\omega').$$

Notice that $E'[\theta] = E'[\theta(., \omega)] \in L^1(\Omega, \mathcal{F}, P)$, and

$$\bar{E}[\theta] (= \int_{\bar{\Omega}} \theta d\bar{P} = \int_{\Omega} E'[\theta(., \omega)] P(d\omega)) = E[E'[\theta]].$$

The driver of our mean-field BSDE is a function $f = f(\omega', \omega, t, y', z', y, z) : \bar{\Omega} \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ which is $\bar{\mathbb{F}}$ -progressively measurable, for all (y', z', y, z) , and satisfies the following assumptions:

(A3) There exists a constant $C \geq 0$ such that, \bar{P} -a.s., for all $t \in [0, T]$, $y_1, y_2, y'_1, y'_2 \in \mathbb{R}$, $z_1, z_2, z'_1, z'_2 \in \mathbb{R}^d$,

$$|f(t, y'_1, z'_1, y_1, z_1) - f(t, y'_2, z'_2, y_2, z_2)| \leq C(|y'_1 - y'_2| + |z'_1 - z'_2| + |y_1 - y_2| + |z_1 - z_2|).$$

(A4) $f(\cdot, 0, 0, 0, 0) \in \mathcal{H}_{\bar{\mathbb{F}}}^2(0, T; \mathbb{R})$.

Remark 3.1. Let $\beta : \Omega \times [0, T] \rightarrow \mathbb{R}$, $\gamma : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ be two square integrable, jointly measurable processes. Then, for our driver, we can define, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $dtP(d\omega)$ -a.e.,

$$\begin{aligned} f^{\beta, \gamma}(\omega, t, y, z) &= E'[f(., \omega, t, \beta'_t, \gamma'_t, y, z)] \\ &= \int_{\Omega} f(\omega', \omega, t, \beta_t(\omega'), \gamma_t(\omega'), y, z) P(d\omega'). \end{aligned}$$

Indeed, we remark that, for all (y, z) , due to our assumptions on the driver f , $(f(., t, \beta'_t, \gamma'_t, y, z)) \in \mathcal{H}_{\bar{\mathbb{F}}}^2(0, T; \mathbb{R})$, and thus $f^{\beta, \gamma}(., ., y, z) \in \mathcal{H}_{\bar{\mathbb{F}}}^2(0, T; \mathbb{R})$. Moreover, with the constant C of assumption (A3), for all $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$, $dtP(d\omega)$ -a.e.,

$$|f^{\beta, \gamma}(\omega, t, y_1, z_1) - f^{\beta, \gamma}(\omega, t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|).$$

Consequently, there is an \mathbb{F} -progressively measurable version of $f^{\beta, \gamma}(., ., y, z)$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, such that $f^{\beta, \gamma}(\omega, t, ., .)$ is $dtP(d\omega)$ -a.e. defined and Lipschitz in (y, z) ; its Lipschitz constant is that introduced in (A3).

We now can state the main result of this section.

Theorem 3.1. Under the assumptions (A3) and (A4), for any random variable $\xi \in L^2(\emptyset, \mathcal{F}_T, P)$, the mean-field BSDE

$$Y_t = \xi + \int_t^T E'[f(s, Y'_s, Z'_s, Y_s, Z_s)] ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad (3.1)$$

has a unique adapted solution

$$(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}_{\bar{\mathbb{F}}}^2(0, T; \mathbb{R}) \times \mathcal{H}_{\bar{\mathbb{F}}}^2(0, T; \mathbb{R}^d).$$

Remark 3.2. We emphasize that, due to our notations, the driving coefficient of (3.1) has to be interpreted as follows

$$\begin{aligned} E'[f(s, Y'_s, Z'_s, Y_s, Z_s)](\omega) &= E'[f(s, Y'_s, Z'_s, Y_s(\omega), Z_s(\omega))] \\ &= \int_{\Omega} f(\omega', \omega, s, Y_s(\omega'), Z_s(\omega'), Y_s(\omega), Z_s(\omega)) P(d\omega'). \end{aligned}$$

Proof. We first introduce a norm on the space $\mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R} \times \mathbb{R}^d)$ which is equivalent to the canonical norm:

$$\|v(\cdot)\|_{\beta} = \left\{ E \int_0^T |v_s|^2 e^{\beta s} ds \right\}^{\frac{1}{2}}, \quad \beta > 0.$$

The parameter β will be specified later.

Step 1: For any $(y, z) \in \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R} \times \mathbb{R}^d)$ there exists a unique solution $(Y, Z) \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^d)$ to the following BSDE:

$$Y_t = \xi + \int_t^T E'[f(s, y'_s, z'_s, Y_s, Z_s)] ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T. \quad (3.2)$$

Indeed, we define $g^{(y, z)}(s, \mu, \nu) = E'[f(s, y'_s, z'_s, \mu, \nu)]$. Then, due to Remark 3.1, $g^{(y, z)}(s, \mu, \nu)$ satisfies (A1) and (A2), and from Lemma 2.1 we know there exists a unique solution $(Y, Z) \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^d)$ to the BSDE (3.2).

Step 2: The result of Step 1 allows to introduce the mapping $(Y, Z) = I[(y, z)] : \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R} \times \mathbb{R}^d) \rightarrow \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R} \times \mathbb{R}^d)$ by the equation

$$Y_t = \xi + \int_t^T E'[f(s, y'_s, z'_s, Y_s, Z_s)] ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T. \quad (3.3)$$

For any $(y^1, z^1), (y^2, z^2) \in \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R} \times \mathbb{R}^d)$ we put $(Y^1, Z^1) = I[(y^1, z^1)]$, $(Y^2, Z^2) = I[(y^2, z^2)]$, $(\hat{y}, \hat{z}) = (y^1 - y^2, z^1 - z^2)$ and $(\hat{Y}, \hat{Z}) = (Y^1 - Y^2, Z^1 - Z^2)$. Then, by applying Itô's formula to $e^{\beta s} |\hat{Y}_s|^2$ and by using that $Y^1, Y^2 \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R})$ we get

$$\begin{aligned} & |\hat{Y}_t|^2 + E[\int_t^T e^{\beta(r-t)} \beta |\hat{Y}_r|^2 dr | \mathcal{F}_t] + E[\int_t^T e^{\beta(r-t)} |\hat{Z}_r|^2 dr | \mathcal{F}_t] \\ &= E[\int_t^T e^{\beta(r-t)} 2\hat{Y}_r (g^{(y^1, z^1)}(r, Y_r^1, Z_r^1) - g^{(y^2, z^2)}(r, Y_r^2, Z_r^2)) dr | \mathcal{F}_t], \quad t \in [0, T]. \end{aligned}$$

From assumption (A3) we obtain

$$\begin{aligned} & \left(\frac{\beta}{2} - 2C - 2C^2 \right) E[\int_0^T e^{\beta r} |\hat{Y}_r|^2 dr] + \frac{1}{2} E[\int_0^T e^{\beta r} |\hat{Z}_r|^2 dr] \\ & \leq \frac{4C^2}{\beta} \{ E[\int_0^T e^{\beta r} |\hat{y}_r|^2 dr] + E[\int_0^T e^{\beta r} |\hat{z}_r|^2 dr] \}. \end{aligned}$$

Thus, taking $\beta = 16C^2 + 4C + 1$ we get

$$E[\int_0^T e^{\beta r} (|\hat{Y}_r|^2 + |\hat{Z}_r|^2) dr] \leq \frac{1}{2} E[\int_0^T e^{\beta r} (|\hat{y}_r|^2 + |\hat{z}_r|^2) dr],$$

that is, $\|(\hat{Y}, \hat{Z})\|_{\beta} \leq \frac{1}{\sqrt{2}} \|(\hat{y}, \hat{z})\|_{\beta}$. Consequently, I is a contraction on $\mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R} \times \mathbb{R}^d)$ endowed with the norm $\|\cdot\|_{\beta}$, and from the contraction mapping theorem we know that there is a unique fixed point $(Y, Z) \in \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R} \times \mathbb{R}^d)$ such that $I(Y, Z) = (Y, Z)$. On the other hand, from Step 1 we already know that if $I(Y, Z) = (Y, Z)$ then $(Y, Z) \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^d)$. \square

Using the notation introduced in Theorem 3.1 we now suppose that, for some $f : \bar{\Omega} \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying (A3) and (A4), the drivers f_i , $i = 1, 2$, are of the form

$$f_i(s, Y_s^{i'}, Z_s^{i'}, Y_s^i, Z_s^i) = f(s, Y_s^{i'}, Z_s^{i'}, Y_s^i, Z_s^i) + \varphi_i(s), \quad \text{d}\bar{P}\text{-a.e.}, \quad i = 1, 2,$$

where $\varphi_i \in \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R})$ and (Y^i, Z^i) is the solution of Mean-Field BSDE (3.1) with data (f_i, ξ_i) , $i = 1, 2$. Then, for arbitrary terminal values ξ_1, ξ_2 belonging to $L^2(\Omega, \mathcal{F}_T, P)$ we have the following

Lemma 3.1. *The difference of the solutions (Y^1, Z^1) and (Y^2, Z^2) of BSDE (3.1) with the data (ξ_1, f_1) and (ξ_2, f_2) , respectively, satisfies the following estimate:*

$$\begin{aligned} & E[|Y_t^1 - Y_t^2|^2] + \frac{1}{2} E\left[\int_t^T e^{\beta(s-t)} (|Y_s^1 - Y_s^2|^2 + |Z_s^1 - Z_s^2|^2) ds\right] \\ & \leq E[e^{\beta(T-t)} |\xi_1 - \xi_2|^2] + \bar{E}\left[\int_t^T e^{\beta(s-t)} |\varphi_1(s) - \varphi_2(s)|^2 ds\right], \quad \text{for all } 0 \leq t \leq T, \end{aligned}$$

where $\beta = 16(1 + C^2)$.

The proof uses a similar argument as that of the proof of Theorem 3.1 and is therefore omitted.

Now we discuss the comparison principle for Mean-Field BSDE. We first give two counterexamples to show that if the driver f depends on z' or f is decreasing with respect to y' we can't get the comparison theorem.

Example 3.1 For $d = 1$ we consider the Mean-Field BSDE (3.1) with time horizon $T = 1$, with driver $f(\omega', \omega, s, y', z', y, z) = -z'$ and two different terminal values $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P)$. Let us denote the associated solutions by (Y^1, Z^1) and (Y^2, Z^2) , respectively. Then,

$$Y_t^i = \xi_i + \int_t^T E[-Z_s^i] ds - \int_t^T Z_s^i dB_s, \quad 0 \leq t \leq 1, \quad i = 1, 2. \quad (3.4)$$

We let $\xi_1 = -(B_1^+)^3$ and define $\tilde{Y}_t^1 = Y_t^1 + \int_t^T E[Z_s^1] ds$. Then (\tilde{Y}^1, Z^1) is the unique solution of the BSDE $\tilde{Y}_t^1 = \xi_1 - \int_t^T Z_s^1 dB_s$, $t \in [0, T]$. Thus, $\tilde{Y}_0^1 = E[\xi_1] = -E[(B_1^+)^3] = -\frac{2}{\sqrt{2\pi}}$. On the other hand, from the Clark-Ocone formula we know that Z^1 is the predictable projection of the Malliavin derivative $(D_t \xi)_{t \in [0, T]}$ of ξ ($D_t \xi$ denotes the Malliavin derivative of ξ at time t ; the interested reader is referred, e.g., to Nualart [11]). This implies $E[Z_t^1] = E[D_t \xi] = E[-3(B_1^+)^2] = -\frac{3}{2}$, $t \in [0, 1]$. Therefore, $Y_0^1 = \tilde{Y}_0^1 - \int_0^T E[Z_s^1] ds = -\frac{2}{\sqrt{2\pi}} + \frac{3}{2} > 0$. Let now $\xi_2 = 0$. Then, obviously, $(Y^2, Z^2) = (0, 0)$. Hence, we have $Y_0^1 > Y_0^2$ although $\xi_1 \leq \xi_2$, P-a.s. and $P\{\xi_1 < \xi_2\} > 0$.

Example 3.2 Let again $d = 1$. We consider Mean-Field BSDE (3.1) driven by the function $f(\omega', \omega, s, y', z', y, z) = -y'$, with time horizon $T = 2$ and two different terminal values $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P)$:

$$Y_t^i = \xi_i + \int_t^T E[-Y_s^i] ds - \int_t^T Z_s^i dB_s, \quad 0 \leq t \leq 2, \quad i = 1, 2. \quad (3.5)$$

By choosing first $\xi_1 = (B_1)^2$ we have $E[Y_t^1] = e^{-(2-t)}$, $t \in [0, 2]$. Furthermore, for $t \in [1, 2]$ we have $Y_t^1 = (B_1)^2 - \int_t^2 e^{-(2-s)} ds = (B_1)^2 - (1 - e^{-(2-t)})$ and $Z_t^1 = 0$, P-a.s. Consequently, $Y_1^1 = (B_1)^2 - (1 - e^{-1}) < 0$ on the set $\{(B_1)^2 < 1 - e^{-1}\}$ which is of strictly positive probability. Finally, for $\xi_2 = 0$ we have the solution $(Y^2, Z^2) = (0, 0)$. Therefore, in our example $P(Y_1^1 < Y_1^2) > 0$, although $\xi_1 > \xi_2$, P-a.s.

The above both examples show that we cannot hope to have the comparison principle between two Mean-Field BSDEs whose drivers depend both on z' or are both decreasing in y' .

Theorem 3.2. (*Comparison Theorem*) Let $f_i = f_i(\bar{\omega}, t, y', z', y, z)$, $i = 1, 2$, be two drivers satisfying the standard assumptions (A3) and (A4). Moreover, we suppose

- (i) One of both coefficients is independent of z' ;
- (ii) One of both coefficients is nondecreasing in y' .

Let $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P)$ and denote by (Y^1, Z^1) and (Y^2, Z^2) the solution of the mean-field BSDE (3.1) with data (ξ_1, f_1) and (ξ_2, f_2) , respectively. Then if $\xi_1 \leq \xi_2$, P -a.s., and $f_1 \leq f_2$, \bar{P} -a.s., it holds that also $Y_t^1 \leq Y_t^2$, $t \in [0, T]$, P -a.s.

Remark 3.3. The conditions (i) and (ii) of Theorem 3.2 are, in particular, satisfied, if they hold for the same driver f_j but also if (i) is satisfied by one driver and (ii) by the other one.

Proof. Without loss of generality, we assume that (i) is satisfied by f_1 and (ii) by f_2 . Then, with the notation $(\bar{Y}, \bar{Z}) := (Y^1 - Y^2, Z^1 - Z^2)$, $\bar{\xi} := \xi_1 - \xi_2$,

$$\bar{Y}_t = \bar{\xi} + \int_t^T E'[f_1(s, Y_s^{1'}, Y_s^1, Z_s^1) - f_2(s, Y_s^{2'}, Z_s^{2'}, Y_s^2, Z_s^2)]ds - \int_t^T \bar{Z}_s dB_s, \quad 0 \leq t \leq T, \quad (3.6)$$

and from Itô's formula applied to an appropriate approximation of $\varphi(y) = (y^+)^2$, $y \in \mathbb{R}$, in which we take after the limit, we obtain

$$\begin{aligned} & E[(\bar{Y}_t^+)^2] + E[\int_t^T I_{\{\bar{Y}_s > 0\}} |\bar{Z}_s|^2 ds] \\ &= 2E[\int_t^T \bar{Y}_s^+ (E'[f_1(s, Y_s^{1'}, Y_s^1, Z_s^1) - f_2(s, Y_s^{2'}, Z_s^{2'}, Y_s^2, Z_s^2)])ds] \\ &\leq 2E[\int_t^T \bar{Y}_s^+ (E'[f_1(s, Y_s^{1'}, Y_s^1, Z_s^1) - f_1(s, Y_s^{1'}, Y_s^2, Z_s^2)])ds] \\ &\quad + 2E[\int_t^T \bar{Y}_s^+ (E'[f_2(s, Y_s^{1'}, Z_s^{2'}, Y_s^2, Z_s^2) - f_2(s, Y_s^{2'}, Z_s^{2'}, Y_s^2, Z_s^2)])ds], \quad t \in [0, T]. \end{aligned}$$

Since $f_2(s, y', z', y, z)$ is nondecreasing in y' we get from (A3)

$$\begin{aligned} & E[\bar{Y}_s^+ (E'[f_1(s, Y_s^{1'}, Y_s^1, Z_s^1) - f_1(s, Y_s^{1'}, Y_s^2, Z_s^2)])] \\ &+ E[\bar{Y}_s^+ (E'[f_2(s, Y_s^{1'}, Z_s^{2'}, Y_s^2, Z_s^2) - f_2(s, Y_s^{2'}, Z_s^{2'}, Y_s^2, Z_s^2)])] \\ &\leq E[\bar{Y}_s^+ (E'[C(Y_s^{1'} - Y_s^{2'})^+ + C|\bar{Y}_s| + C|\bar{Z}_s|])] \\ &= C(E[\bar{Y}_s^+])^2 + CE[(\bar{Y}_s^+)^2] + CE[\bar{Y}_s^+ |\bar{Z}_s|] \\ &\leq (2C + C^2)E[(\bar{Y}_s^+)^2] + \frac{1}{4}E[I_{\{\bar{Y}_s > 0\}} |\bar{Z}_s|^2], \quad s \in [0, T]. \end{aligned}$$

Consequently,

$$E[(\bar{Y}_t^+)^2] + \frac{1}{2}E[\int_t^T I_{\{\bar{Y}_s > 0\}} |\bar{Z}_s|^2 ds] \leq (4C + 2C^2) \int_t^T E[(\bar{Y}_s^+)^2]ds, \quad t \in [0, T],$$

from where we can conclude with the help of Gronwall's Lemma that $Y_t^1 - Y_t^2 = \bar{Y}_t \leq 0$, $t \in [0, T]$, P -a.s. \square

We also have a converse comparison theorem.

Theorem 3.3. (*Converse Comparison Theorem*) We retake the assumptions of Theorem 3.2 and suppose that, additionally, for some $t \in [0, T]$, $Y_t^1 = Y_t^2$, P -a.s. Then

- (i) $Y_s^1 = Y_s^2$, $s \in [t, T]$, P -a.s., and
- (ii) If f_2 satisfies (ii) of Theorem 3.2 then $E'[f_1(s, Y_s^{1'}, Z_s^{1'}, Y_s^2, Z_s^2)] = E'[f_2(s, Y_s^{2'}, Z_s^{2'}, Y_s^2, Z_s^2)]$, $dsdP$ -a.e. on $[t, T]$, and if f_1 satisfies (ii) then the symmetric result holds.

Proof. We use the notation introduced in the proof of Theorem 3.2 and suppose again that f_1 satisfies (i) and f_2 (ii). Then, from the Lipschitz property of f_1 , there exist some $\bar{\mathbb{F}}$ -progressively measurable, bounded processes α, β , defined over $\bar{\Omega} \times [0, T]$, such that

$$f_1(s, Y_s^{1'}, Y_s^1, Z_s^1) - f_1(s, Y_s^{1'}, Y_s^2, Z_s^2) = \alpha_s \bar{Y}_s + \beta_s \bar{Z}_s, \quad s \in [0, T],$$

and since Y^1, Z^1, Y^2, Z^2 don't depend on ω' ,

$$E'[f_1(s, Y_s^{1'}, Y_s^1, Z_s^1) - f_1(s, Y_s^{1'}, Y_s^2, Z_s^2)] = E'[\alpha_s] \bar{Y}_s + E'[\beta_s] \bar{Z}_s, \quad s \in [0, T].$$

Thus, from Itô's formula,

$$\begin{aligned} \bar{Y}_t = & \exp\left\{\int_t^T E'[\alpha_r] dr\right\} \bar{\xi} + \int_t^T \exp\left\{\int_t^s E'[\alpha_r] dr\right\} E'[f_1(s, Y_s^{1'}, Y_s^2, Z_s^2) - f_2(s, Y_s^{2'}, Z_s^{2'}, Y_s^2, Z_s^2)] ds \\ & - \int_t^T \exp\left\{\int_t^s E'[\alpha_r] dr\right\} \bar{Z}_s d\tilde{B}_s, \quad \text{P-a.s.}, \end{aligned}$$

where $\tilde{B}_s = B_s - \int_0^s E'[\beta_r] dr$, $s \in [0, T]$. It is well known that $\tilde{B} = (\tilde{B}_s)$ is an (\mathbb{F}, \tilde{P}) -Brownian motion with $\tilde{P} := \exp\left\{\int_0^T E'[\beta_s] dB_s - \frac{1}{2} \int_0^T |E'[\beta_s]|^2 ds\right\} P$. From the boundedness of α and β we then deduce easily that $\int_t^T \exp\left\{\int_t^s E'[\alpha_r] dr\right\} \bar{Z}_s d\tilde{B}_s$ is an (\mathbb{F}, \tilde{P}) -martingale increment. Consequently,

$$\begin{aligned} 0 = \bar{Y}_t = & E_{\tilde{P}}[\exp\left\{\int_t^u E'[\alpha_r] dr\right\} \bar{Y}_u | \mathcal{F}_t] \\ & + E_{\tilde{P}}[\int_t^u \exp\left\{\int_t^s E'[\alpha_r] dr\right\} E'[f_1(s, Y_s^{1'}, Y_s^2, Z_s^2) - f_2(s, Y_s^{2'}, Z_s^{2'}, Y_s^2, Z_s^2)] ds | \mathcal{F}_t], \\ & \text{P-a.s., for all } t \leq u \leq T. \end{aligned}$$

To conclude it suffices now to recall that, due to Theorem 3.2, $\bar{Y}_u = Y_u^1 - Y_u^2 \leq 0$, P-a.s. and

$$f_1(s, Y_s^{1'}, Y_s^2, Z_s^2) \leq f_2(s, Y_s^{1'}, Z_s^{2'}, Y_s^2, Z_s^2) \leq f_2(s, Y_s^{2'}, Z_s^{2'}, Y_s^2, Z_s^2), \quad \text{dsd}\bar{P}\text{-a.e. on } \bar{\Omega} \times [t, T].$$

□

4 Mean-Field Stochastic Differential Equations

We consider measurable functions $b : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ which are supposed to satisfy the following conditions:

- (i) $b(\cdot, 0, 0)$ and $\sigma(\cdot, 0, 0)$ are $\bar{\mathbb{F}}$ -progressively measurable continuous processes and there exists some constant $C > 0$ such that

$$|b(t, x', x)| + |\sigma(t, x', x)| \leq C(1 + |x|), \quad \text{a.s., for all } 0 \leq t \leq T, \quad x, x' \in \mathbb{R}^n;$$

- (ii) b and σ are Lipschitz in x, x' , i.e., there is some constant $C > 0$ such that

$$\begin{aligned} |b(t, x'_1, x_1) - b(t, x'_2, x_2)| + |\sigma(t, x'_1, x_1) - \sigma(t, x'_2, x_2)| \leq C(|x'_1 - x'_2| + |x_1 - x_2|), \quad \text{a.s.,} \\ \text{for all } 0 \leq t \leq T, \quad x_1, x'_1, x_2, x'_2 \in \mathbb{R}^n. \end{aligned}$$

(H4.1)

We now study the following SDE parameterized by the initial condition $(t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$:

$$\begin{cases} dX_s^{t, \zeta} = E'[b(s, (X_s^{t, \zeta})', X_s^{t, \zeta})] ds + E'[\sigma(s, (X_s^{t, \zeta})', X_s^{t, \zeta})] dB_s, & s \in [t, T], \\ X_t^{t, \zeta} = \zeta. \end{cases} \quad (4.1)$$

Recall that

$$E'[b(s, (X_s^{t, \zeta})', X_s^{t, \zeta})](\omega) = \int_{\Omega} b(\omega', \omega, s, X_s^{t, \zeta}(\omega'), X_s^{t, \zeta}(\omega)) P(d\omega'), \quad \omega \in \Omega.$$

Theorem 4.1. *Under the assumption (H4.1) SDE (4.1) has a unique strong solution.*

The proof is made in two steps like that of Theorem 3.1 and it uses standard arguments for forward SDEs. Since the proof is straight-forward we prefer to omit it.

Remark 4.1. *From standard arguments we also can get that, for any $p \geq 2$, there exists $C_p \in \mathbb{R}$ such that, for all $t \in [0, T]$ and $\zeta, \zeta' \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,*

$$\begin{aligned} E[\sup_{t \leq s \leq T} |X_s^{t, \zeta} - X_s^{t, \zeta'}|^p | \mathcal{F}_t] &\leq C_p |\zeta - \zeta'|^p, \quad a.s., \\ E[\sup_{t \leq s \leq T} |X_s^{t, \zeta}|^p | \mathcal{F}_t] &\leq C_p (1 + |\zeta|^p), \quad a.s., \\ E[\sup_{t \leq s \leq t + \delta} |X_s^{t, \zeta} - \zeta|^p | \mathcal{F}_t] &\leq C_p (1 + |\zeta|^p) \delta^{\frac{p}{2}}, \end{aligned} \quad (4.2)$$

P -a.s., for all $\delta > 0$ with $t + \delta \leq T$.

These in the classical case well-known standard estimates can be consulted, for instance, in Ikeda, Watanabe [8], pp.166-168 and also in Karatzas, Shreve [9], pp.289-290. We also emphasize that the constant C_p in (4.2) only depends on the Lipschitz and the growth constants of b and σ .

5 Decoupled Mean-Field Forward-Backward SDE and Related DPP

In this section we study a decoupled Mean-Field forward-backward SDE and its relation with PDEs. Given continuous functions $b : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ which are supposed to satisfy the conditions (i) and (ii) of (H4.1) and an arbitrary $x_0 \in \mathbb{R}^n$, we consider the following SDE parameterized by the initial condition $(t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$:

$$\begin{cases} dX_s^{t, \zeta} &= E'[b(s, (X_s^{0, x_0})', X_s^{t, \zeta})] ds + E'[\sigma(s, (X_s^{0, x_0})', X_s^{t, \zeta})] dB_s, \quad s \in [t, T], \\ X_t^{t, \zeta} &= \zeta. \end{cases} \quad (5.1)$$

Under the assumption (H4.1), SDE (5.1) has a unique strong solution. Indeed, from Theorem 4.1 we first deduce the existence and uniqueness of the process $X^{0, x_0} \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}^n)$ to the Mean-Field SDE (5.1). Once knowing X^{0, x_0} , SDE (5.1) becomes a classical equation with the coefficients $\tilde{b}(\omega, s, x) = E'[b(\omega', \omega, s, X_s^{0, x_0}(\omega'), x)]$ and $\tilde{\sigma}(\omega, s, x) = E'[\sigma(\omega', \omega, s, X_s^{0, x_0}(\omega'), x)]$. Combining estimate (4.2) for $(t, \zeta) = (0, x_0)$ with standard arguments for SDEs we obtain (4.2) also for equation (5.1).

Let now be given two real-valued functions $f(t, x', x, y', y, z)$ and $\Phi(x', x)$ which shall satisfy the following conditions:

- (i) $\Phi : \bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is an $\bar{\mathcal{F}}_T \otimes \mathcal{B}(\mathbb{R}^n)$ -measurable random variable and $f : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable process such that $f(\cdot, x', x, y', y, z)$ is $\bar{\mathbb{F}}$ -adapted, for all $(x', x, y', y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$;
- (ii) There exists a constant $C > 0$ such that
$$\begin{aligned} &|f(t, x'_1, x_1, y'_1, y_1, z_1) - f(t, x'_2, x_2, y'_2, y_2, z_2)| + |\Phi(x'_1, x_1) - \Phi(x'_2, x_2)| \\ &\leq C(|x'_1 - x'_2| + |x_1 - x_2| + |y'_1 - y'_2| + |y_1 - y_2| + |z_1 - z_2|), \quad a.s., \\ &\text{for all } 0 \leq t \leq T, \quad x_1, x'_1, x_2, x'_2 \in \mathbb{R}^n, \quad y_1, y'_1, y_2, y'_2 \in \mathbb{R} \text{ and } z_1, z_2 \in \mathbb{R}^d; \end{aligned}$$
- (iii) f and Φ satisfy a linear growth condition, i.e., there exists some $C > 0$ such that, a.s., for all $x', x \in \mathbb{R}^n$,
$$|f(t, x', x, 0, 0, 0)| + |\Phi(x', x)| \leq C(1 + |x| + |x'|); \quad (\text{H5.1})$$
- (iv) $f(\bar{\omega}, t, x', x, y', y, z)$ is continuous in t for all (x', x, y', y, z) , $P(d\bar{\omega})$ -a.s.;
- (v) $f(t, x', x, y', y, z)$ is nondecreasing with respect to y' .

We consider the following BSDE:

$$\begin{cases} -dY_s^{t,\zeta} &= E'[f(s, (X_s^{0,x_0})', X_s^{t,\zeta}, (Y_s^{0,x_0})', Y_s^{t,\zeta}, Z_s^{t,\zeta})]ds - Z_s^{t,\zeta}dB_s, \quad s \in [t, T], \\ Y_T^{t,\zeta} &= E'[\Phi((X_T^{0,x_0})', X_T^{t,\zeta})]. \end{cases} \quad (5.2)$$

We first consider the equation (5.2) for $(t, \zeta) = (0, x_0)$: We know from Theorem 3.1 that there exists a unique solution $(Y^{0,x_0}, Z^{0,x_0}) \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^d)$ to the Mean-Field BSDE (5.2). Once we have (Y^{0,x_0}, Z^{0,x_0}) , equation (5.2) becomes a classical BSDE whose coefficients $\tilde{f}(\omega, s, X_s^{t,\zeta}, y, z) = E'[f(., \omega, s, (X_s^{0,x_0})', X_s^{t,\zeta}, (Y_s^{0,x_0})', y, z)]$ satisfies the assumptions (A1) and (A2), and $\tilde{\Phi}(\omega, X_T^{t,\zeta}(\omega)) = E'[\Phi(., \omega, (X_T^{0,x_0})', X_T^{t,\zeta})] \in L^2(\Omega, \mathcal{F}_T, P)$. Thus, from Lemma 2.1 we know that there exists a unique solution $(Y^{t,\zeta}, Z^{t,\zeta}) \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^d)$ to equation (5.2).

By combining classical BSDE estimates (see, e.g., Proposition 4.1 in Peng [14]; or Proposition 4.1 in El Karoui, Peng and Quenez [7]) with the techniques presented above we see that there exists a constant C such that, for any $t \in [0, T]$ and $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have the following estimates:

$$\begin{aligned} \text{(i)} & E[\sup_{t \leq s \leq T} |Y_s^{t,\zeta}|^2 + \int_t^T |Z_s^{t,\zeta}|^2 ds | \mathcal{F}_t] \leq C(1 + |\zeta|^2), \quad a.s.; \\ \text{(ii)} & E[\sup_{t \leq s \leq T} |Y_s^{t,\zeta} - Y_s^{t,\zeta'}|^2 + \int_t^T |Z_s^{t,\zeta} - Z_s^{t,\zeta'}|^2 ds | \mathcal{F}_t] \leq C|\zeta - \zeta'|^2, \quad a.s. \end{aligned}$$

In particular,

$$\begin{aligned} \text{(iii)} & |Y_t^{t,\zeta}| \leq C(1 + |\zeta|), \quad a.s.; \\ \text{(iv)} & |Y_t^{t,\zeta} - Y_t^{t,\zeta'}| \leq C|\zeta - \zeta'|, \quad a.s. \end{aligned} \quad (5.3)$$

Here the constant $C > 0$ depends only on the Lipschitz and the growth constants of b, σ, f and Φ .

Let us now introduce the random field:

$$u(t, x) = Y_s^{t,x}|_{s=t}, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad (5.4)$$

where $Y^{t,x}$ is the solution of BSDE (5.2) with $x \in \mathbb{R}^n$ at the place of $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$.

As a consequence of (5.3) we immediately have that, for all $t \in [0, T]$, P-a.s.,

$$\begin{aligned} \text{(i)} & |u(t, x) - u(t, y)| \leq C|x - y|, \quad \text{for all } x, y \in \mathbb{R}^n; \\ \text{(ii)} & |u(t, x)| \leq C(1 + |x|), \quad \text{for all } x \in \mathbb{R}^n. \end{aligned} \quad (5.5)$$

Remark 5.1. In the general situation u is an adapted random function, that is, for any $x \in \mathbb{R}^n$, $u(\cdot, x)$ is an \mathbb{F} -adapted real-valued process. Indeed, recall that b, σ and f all are $\bar{\mathbb{F}}$ -adapted random functions while Φ is $\bar{\mathcal{F}}_T$ -measurable. However, if the functions b, σ, f and Φ are deterministic it is well known that also u is a deterministic function of (t, x) (see, e.g., Proposition 2.4 in Peng [14]).

From now on, let us suppose that

$$\begin{aligned} \text{(vi)} & \text{ The coefficients } b, \sigma, f \text{ and } \Phi \text{ are deterministic, i.e., independent of} \\ & (\omega', \omega) \in \bar{\Omega} = \Omega \times \Omega. \end{aligned} \quad (\text{H5.2})$$

The function u and the random field $Y^{t,\zeta}$, $(t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, are related by the following theorem.

Theorem 5.1. *Under the assumptions (H4.1) and (H5.1), for any $t \in [0, T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have*

$$u(t, \zeta) = Y_t^{t, \zeta}, \quad P\text{-a.s.} \quad (5.6)$$

Indeed, once X^{0, x_0} and Y^{0, x_0} are determined, the coefficients of SDE (5.1) and BSDE (5.2) are well-determined, deterministic, depend only on (t, x) and (t, x, y, z) , respectively, and satisfy the standard growth and Lipschitz conditions. Hence, Theorem 5.1 is a consequence of the corresponding result in Peng [14] (Theorem 4.7) (or, see also Theorem 6.1 in Buckdahn and Li [4]).

We now discuss (the generalized) DPP for our FBSDE (5.1), (5.2). For this end we have to define the family of (backward) semigroups associated with BSDE (5.2). This notion of stochastic backward semigroup was first introduced by Peng [14] and originally applied to study the DPP for stochastic control problems. Our approach extends Peng's ideas to the framework of Mean-Field FBSDE. However, we change the definition of the stochastic backward semigroup to simplify the proof of the existence of a viscosity solution of the associated PDE.

Given the initial data (t, x) , a positive number $\delta \leq T - t$ and a real-valued random variable $\eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; \mathbb{R})$, we put

$$G_{s, t+\delta}^{t, x}[\eta] := \tilde{Y}_s^{t, x}, \quad s \in [t, t + \delta], \quad (5.7)$$

where the couple $(\tilde{Y}_s^{t, x}, \tilde{Z}_s^{t, x})_{t \leq s \leq t+\delta}$ is the solution of the following BSDE with the time horizon $t + \delta$:

$$\begin{cases} -d\tilde{Y}_s^{t, x} = E'[f(s, (X_s^{0, x_0})', X_s^{t, x}, (Y_s^{0, x_0})', Y_s^{t, x}, \tilde{Z}_s^{t, x})]ds - \tilde{Z}_s^{t, x}dB_s, & s \in [t, t + \delta], \\ \tilde{Y}_{t+\delta}^{t, x} = \eta; \end{cases} \quad (5.8)$$

here $X^{t, x}$ is the solution of SDE (5.1) and $Y^{t, x}$ is the solution of BSDE (5.2). Then, obviously, for the solution $(Y^{t, x}, Z^{t, x})$ of BSDE (5.2) we have

$$G_{t, T}^{t, x}[\tilde{\Phi}(X_T^{t, x})] = G_{t, t+\delta}^{t, x}[Y_{t+\delta}^{t, x}], \quad 0 \leq t < t + \delta \leq T. \quad (5.9)$$

Let us point out that in difference to Peng's definition of the backward semigroup the driver of our BSDE depends on the processes Y^{0, x_0} and $Y^{t, x}$ given by BSDE (5.2). The choice to let the driver f depend on these processes and not on $\tilde{Y}^{0, x_0}, \tilde{Y}^{t, x}$ will simplify the proof of the existence theorem for the associated nonlocal PDEs in Section 6.

Moreover, we have the following DPP

$$\begin{aligned} u(t, x) &= Y_t^{t, x} = G_{t, T}^{t, x}[\tilde{\Phi}(X_T^{t, x})] = G_{t, t+\delta}^{t, x}[Y_{t+\delta}^{t, x}] \\ &= G_{t, t+\delta}^{t, x}[u(t + \delta, X_{t+\delta}^{t, x})], \end{aligned} \quad (5.10)$$

whose simple form explains by the fact that our stochastic evolution system doesn't depend on a control. Here, for the latter relation we have used, that due to the uniqueness of the solution of SDE (5.1) and of BSDE (5.2), $Y_{t+\delta}^{t, x} = Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t, x}}$, P-a.s., so that then from Theorem 5.1 it follows that $Y_{t+\delta}^{t, x} = u(t + \delta, X_{t+\delta}^{t, x})$, P-a.s.

Remark 5.2. *If f is independent of (y, z) it holds that*

$$G_{s, t+\delta}^{t, x}[\eta] = E[\eta + \int_s^{t+\delta} E'[f(r, (X_r^{0, x_0})', X_r^{t, x}, (Y_r^{0, x_0})')]dr | \mathcal{F}_s], \quad s \in [t, t + \delta], \quad P\text{-a.s.}$$

In (5.5) we have already seen that the value function $u(t, x)$ is Lipschitz continuous in x , uniformly in t . Relation (5.10) now allows also to study the continuity property of $u(t, x)$ in t .

Theorem 5.2. *Let us suppose that the assumptions (H4.1), (H5.1) and (H5.2) hold. Then the value function $u(t, x)$ is $\frac{1}{2}$ -Hölder continuous in t , locally uniformly with respect to x : There exists a constant C such that, for every $x \in \mathbb{R}^n$, $t, t' \in [0, T]$,*

$$|u(t, x) - u(t', x)| \leq C(1 + |x|)|t - t'|^{\frac{1}{2}}.$$

Proof. Let $(t, x) \in [0, T] \times \mathbb{R}^n$ and $\delta > 0$ be arbitrarily given such that $0 < \delta \leq T - t$. Our objective is to prove the following inequality by using DPP:

$$-C(1 + |x|)\delta^{\frac{1}{2}} \leq u(t, x) - u(t + \delta, x) \leq C(1 + |x|)\delta^{\frac{1}{2}}. \quad (5.11)$$

From it we obtain immediately that u is $\frac{1}{2}$ -Hölder continuous in t . We will only check the second inequality in (5.11), the first one can be shown in a similar way. To this end we note that due to (5.10),

$$u(t, x) - u(t + \delta, x) = I_\delta^1 + I_\delta^2, \quad (5.12)$$

where

$$\begin{aligned} I_\delta^1 &:= G_{t, t+\delta}^{t, x}[u(t + \delta, X_{t+\delta}^{t, x})] - G_{t, t+\delta}^{t, x}[u(t + \delta, x)], \\ I_\delta^2 &:= G_{t, t+\delta}^{t, x}[u(t + \delta, x)] - u(t + \delta, x). \end{aligned}$$

From Lemmata 2.3 and 3.1 and the estimate (5.5) we obtain that,

$$\begin{aligned} |I_\delta^1| &\leq [CE(|u(t + \delta, X_{t+\delta}^{t, x}) - u(t + \delta, x)|^2 | \mathcal{F}_t)]^{\frac{1}{2}} \\ &\leq [CE(|X_{t+\delta}^{t, x} - x|^2 | \mathcal{F}_t)]^{\frac{1}{2}}, \end{aligned}$$

and since $E[|X_{t+\delta}^{t, x} - x|^2 | \mathcal{F}_t] \leq C(1 + |x|^2)\delta$ we deduce that $|I_\delta^1| \leq C(1 + |x|)\delta^{\frac{1}{2}}$. From the definition of $G_{t, t+\delta}^{t, x}[\cdot]$ (see (5.7)) we know that the second term I_δ^2 can be written as

$$\begin{aligned} I_\delta^2 &= E[u(t + \delta, x) + \int_t^{t+\delta} E'[f(s, (X_s^{0, x_0})', X_s^{t, x}, (Y_s^{0, x_0})', Y_s^{t, x}, \tilde{Z}_s^{t, x})] ds \\ &\quad - \int_t^{t+\delta} \tilde{Z}_s^{t, x} dB_s | \mathcal{F}_t] - u(t + \delta, x) \\ &= E[\int_t^{t+\delta} E'[f(s, (X_s^{0, x_0})', X_s^{t, x}, (Y_s^{0, x_0})', Y_s^{t, x}, \tilde{Z}_s^{t, x})] ds | \mathcal{F}_t]. \end{aligned}$$

Then, with the help of the Schwartz inequality, and the estimates (4.2), (5.3)-(i) for the BSDEs (5.2) and (5.8) (with $\eta = u(t + \delta, x)$) and (5.5) we have

$$\begin{aligned} |I_\delta^2| &\leq \delta^{\frac{1}{2}} E[\int_t^{t+\delta} |E'[f(s, (X_s^{0, x_0})', X_s^{t, x}, (Y_s^{0, x_0})', Y_s^{t, x}, \tilde{Z}_s^{t, x})]|^2 ds | \mathcal{F}_t]^{\frac{1}{2}} \\ &\leq \delta^{\frac{1}{2}} E[\int_t^{t+\delta} (|E'[f(s, (X_s^{0, x_0})', X_s^{t, x}, (Y_s^{0, x_0})', 0, 0)]| + C|Y_s^{t, x}| + C|\tilde{Z}_s^{t, x}|)^2 ds | \mathcal{F}_t]^{\frac{1}{2}} \\ &\leq C\delta^{\frac{1}{2}} E[\int_t^{t+\delta} (1 + |X_s^{t, x}| + |Y_s^{t, x}| + |\tilde{Z}_s^{t, x}|)^2 ds | \mathcal{F}_t]^{\frac{1}{2}} \\ &\leq C(1 + |x|)\delta^{\frac{1}{2}}. \end{aligned}$$

Hence, from (5.12), we get the second inequality of (5.11)

$$u(t, x) - u(t + \delta, x) \leq C(1 + |x|)\delta^{\frac{1}{2}}.$$

The proof is complete. □

6 Viscosity Solution Of PDE: Existence Theorem

In this section we consider the following PDE

$$\begin{cases} \frac{\partial}{\partial t}u(t, x) + Au(t, x) + E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), Du(t, x).E[\sigma(t, X_t^{0,x_0}, x)])] = 0, \\ (t, x) \in [0, T) \times \mathbb{R}^n, \\ u(T, x) = E[\Phi(X_T^{0,x_0}, x)], \quad x \in \mathbb{R}^n, \end{cases} \quad (6.1)$$

with

$$Au(t, x) := \frac{1}{2} \text{tr}(E[\sigma(t, X_t^{0,x_0}, x)]E[\sigma(t, X_t^{0,x_0}, x)]^T D^2u(t, x)) + Du(t, x).E[b(t, X_t^{0,x_0}, x)].$$

Here the functions b, σ, f and Φ are supposed to satisfy (H4.1), (H5.1) and (H5.2), respectively, and X^{0,x_0} is the solution of the Mean-Field SDE (5.1).

We attract the reader's attention to the fact that, since

$$\begin{aligned} & E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), Du(t, x).E[\sigma(t, X_t^{0,x_0}, x)])] \\ = & \int_{\mathbb{R}^n} f(t, x', x, u(t, x'), u(t, x), Du(t, x).E[\sigma(t, X_t^{0,x_0}, x)]) P_{X_t^{0,x_0}}(dx'), \end{aligned}$$

the above equation is in fact a nonlocal PDE.

In this section we want to prove that the value function $u(t, x)$ introduced by (5.4) is the viscosity solution of equation (6.1). For this we extend Peng's BSDE approach [14] developed in the framework of stochastic control theory to that of the Mean-Field FBSDE. The difficulties related with this extension come from the fact that now, contrarily to the framework of stochastic control theory studied by Peng, we have to do with nonlocal PDEs. Moreover, in difference to [3] the nonlocal term is not generated by a diffusion process with jumps. This fact is the source of difficulties mainly in the proof of the uniqueness of the viscosity solution (given in the next section) which are different to those in [3]. Let us first recall the definition of a viscosity solution of equation (6.1). The reader more interested in viscosity solutions is referred to Crandall, Ishii and Lions [6].

Definition 6.1. *A real-valued continuous function $u \in C_p([0, T] \times \mathbb{R}^n)$ is called*

(i) *a viscosity subsolution of equation (6.1) if, firstly, $u(T, x) \leq E[\Phi(X_T^{0,x_0}, x)]$, for all $x \in \mathbb{R}^n$, and if, secondly, for all functions $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$ and $(t, x) \in [0, T) \times \mathbb{R}^n$ such that $u - \varphi$ attains its local maximum at (t, x) ,*

$$\begin{aligned} & \frac{\partial}{\partial t}\varphi(t, x) + D\varphi(t, x).E[b(t, X_t^{0,x_0}, x)] + \frac{1}{2}\text{tr}(E[\sigma(t, X_t^{0,x_0}, x)]E[\sigma(t, X_t^{0,x_0}, x)]^T D^2\varphi(t, x)) \\ & + E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), D\varphi(t, x).E[\sigma(t, X_t^{0,x_0}, x)])] \geq 0; \end{aligned}$$

(ii) *a viscosity supersolution of equation (6.1) if, firstly, $u(T, x) \geq E[\Phi(X_T^{0,x_0}, x)]$, for all $x \in \mathbb{R}^n$, and if, secondly, for all functions $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$ and $(t, x) \in [0, T) \times \mathbb{R}^n$ such that $u - \varphi$ attains its local minimum at (t, x) ,*

$$\begin{aligned} & \frac{\partial}{\partial t}\varphi(t, x) + D\varphi(t, x).E[b(t, X_t^{0,x_0}, x)] + \frac{1}{2}\text{tr}(E[\sigma(t, X_t^{0,x_0}, x)]E[\sigma(t, X_t^{0,x_0}, x)]^T D^2\varphi(t, x)) \\ & + E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), D\varphi(t, x).E[\sigma(t, X_t^{0,x_0}, x)])] \leq 0; \end{aligned}$$

(iii) *a viscosity solution of equation (6.1) if it is both a viscosity sub- and a supersolution of equation (6.1).*

Remark 6.1. (i) $C_p([0, T] \times \mathbb{R}^n) = \{u \in C([0, T] \times \mathbb{R}^n) : \text{There exists some constant } p > 0 \text{ such that } \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} \frac{|u(t,x)|}{1+|x|^p} < +\infty\}$.

(ii) Usually the definition of a viscosity solution is given for test fields φ of class $C^{1,2}$. However, it can be shown that it is sufficient to work with test functions from $C_{l,b}^3([0, T] \times \mathbb{R}^n)$. The space $C_{l,b}^3([0, T] \times \mathbb{R}^n)$ denotes the set of the real-valued functions that are continuously differentiable up to the third order and whose derivatives of order from 1 to 3 are bounded.

We now can state the main statement of this section.

Theorem 6.1. Under the assumptions (H4.1), (H5.1) and (H5.2) the function $u(t, x)$ defined by (5.4) is a viscosity solution of equation (6.1).

The proof of the theorem uses the BSDE method of Peng [14]. However, it is simplified by the specific choice of our stochastic backward semigroup. For the proof of this theorem we need four auxiliary lemmata. To abbreviate notations we put, for some arbitrarily chosen but fixed $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$,

$$F(s, x, z) = \frac{\partial}{\partial s} \varphi(s, x) + \frac{1}{2} \text{tr}(\tilde{\sigma} \tilde{\sigma}^T(s, x) D^2 \varphi) + D\varphi \cdot \tilde{b}(s, x) + \tilde{f}^u(s, x, z + D\varphi(s, x) \cdot \tilde{\sigma}(s, x)), \quad (6.2)$$

where

$$\begin{aligned} \tilde{\sigma}(s, x) &= E[\sigma(s, X_s^{0,x_0}, x)], \quad \tilde{b}(s, x) = E[b(s, X_s^{0,x_0}, x)]; \\ \tilde{f}^u(s, x, z) &= E[f(s, X_s^{0,x_0}, x, u(s, X_s^{0,x_0}), u(s, x), z)], \end{aligned}$$

$(s, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^d$, and we consider the following BSDE defined on the interval $[t, t + \delta]$ ($0 < \delta \leq T - t$):

$$\begin{cases} -dY_s^1 = F(s, X_s^{t,x}, Z_s^1) ds - Z_s^1 dB_s, \\ Y_{t+\delta}^1 = 0, \end{cases} \quad (6.3)$$

where the process $X^{t,x}$ has been introduced by equation (5.1).

Remark 6.2. It's not hard to check that $F(s, X_s^{t,x}, z)$ satisfies (A1) and (A2). Thus, due to Lemma 2.1 equation (6.3) has a unique solution.

We can characterize the solution process Y^1 as follows:

Lemma 6.1. For every $s \in [t, t + \delta]$, we have the following relationship:

$$Y_s^1 = G_{s,t+\delta}^{t,x}[\varphi(t + \delta, X_{t+\delta}^{t,x})] - \varphi(s, X_s^{t,x}), \quad P\text{-a.s.} \quad (6.4)$$

Proof. We recall that $G_{s,t+\delta}^{t,x}[\varphi(t + \delta, X_{t+\delta}^{t,x})]$ is defined with the help of the solution of the BSDE

$$\begin{cases} -d\tilde{Y}_s = E'[f(s, (X_s^{0,x_0})', X_s^{t,x}, (Y_s^{0,x_0})', Y_s^{t,x}, \tilde{Z}_s)] ds - \tilde{Z}_s dB_s, \quad s \in [t, t + \delta], \\ \tilde{Y}_{t+\delta} = \varphi(t + \delta, X_{t+\delta}^{t,x}), \end{cases}$$

by the following formula:

$$G_{s,t+\delta}^{t,x}[\varphi(t + \delta, X_{t+\delta}^{t,x})] = \tilde{Y}_s, \quad s \in [t, t + \delta] \quad (6.5)$$

(see (5.7)). We also recall that $E'[f(s, (X_s^{0,x_0})', X_s^{t,x}, (Y_s^{0,x_0})', Y_s^{t,x}, \tilde{Z}_s)] = \tilde{f}^u(s, X_s^{t,x}, \tilde{Z}_s)$. Therefore, we only need to prove that $\tilde{Y}_s - \varphi(s, X_s^{t,x}) \equiv Y_s^1$. This result can be obtained easily by applying Itô's formula to $\varphi(s, X_s^{t,x})$. Indeed, we get that the stochastic differentials of $\tilde{Y}_s - \varphi(s, X_s^{t,x})$ and Y_s^1 coincide, while at the terminal time $t + \delta$, $\tilde{Y}_{t+\delta} - \varphi(t + \delta, X_{t+\delta}^{t,x}) = 0 = Y_{t+\delta}^1$. So the proof is complete. \square

We now introduce the deterministic function

$$Y_s^2 = \int_s^{t+\delta} F(r, x, 0) dr, \quad s \in [t, t + \delta].$$

Obviously, the couple $(Y^2, Z^2) = (Y^2, 0)$ is the unique solution of the following (deterministic) BSDE in which the driving process $X^{t,x}$ is replaced by its deterministic initial value x :

$$\begin{cases} -dY_s^2 = F(s, x, Z_s^2)ds - Z_s^2 dB_s, \\ Y_{t+\delta}^2 = 0, \quad s \in [t, t + \delta]. \end{cases} \quad (6.6)$$

The following lemma will allow us to neglect the difference $|Y_t^1 - Y_t^2|$ for sufficiently small $\delta > 0$.

Lemma 6.2. *We have*

$$|Y_t^1 - Y_t^2| \leq C\delta^{\frac{3}{2}}, \quad P\text{-a.s.} \quad (6.7)$$

Proof. We recall that from (4.2) with $\zeta = x$ it follows that there is some constant C_p depending on p, x , but not on $\delta > 0$, such that

$$E[\sup_{t \leq s \leq t+\delta} |X_s^{t,x} - x|^p | \mathcal{F}_t] \leq C_p \delta^{\frac{p}{2}}. \quad (6.8)$$

We now apply Lemma 2.3 combined with (6.8) to the equations (6.3) and (6.6). For this we set in Lemma 2.3:

$$\begin{aligned} \xi_1 = \xi_2 = 0, \quad g(s, y, z) &= g(s, z) = F(s, X_s^{t,x}, z), \\ \varphi_1(s) &= 0, \quad \varphi_2(s) = F(s, x, Z_s^2) - F(s, X_s^{t,x}, Z_s^2). \end{aligned}$$

Obviously, the function g is Lipschitz with respect to z , and $|\varphi_2(s)| \leq C(1 + |x|^2)(|X_s^{t,x} - x| + |X_s^{t,x} - x|^3)$, for $s \in [t, t + \delta]$, $(t, x) \in [0, T] \times \mathbb{R}^n$. Thus, with the notation $\rho_0(r) = (1 + |x|^2)(r + r^3)$, $r \geq 0$, we have

$$\begin{aligned} & E[\int_t^{t+\delta} |Z_s^1 - Z_s^2|^2 ds | \mathcal{F}_t] \\ & \leq CE[\int_t^{t+\delta} \rho_0^2(|X_s^{t,x} - x|) ds | \mathcal{F}_t] \\ & \leq C\delta E[\sup_{t \leq s \leq t+\delta} \rho_0^2(|X_s^{t,x} - x|) | \mathcal{F}_t] \\ & \leq C\delta^2. \end{aligned}$$

Therefore,

$$\begin{aligned} & |Y_t^1 - Y_t^2| = |E[(Y_t^1 - Y_t^2) | \mathcal{F}_t]| \\ & = |E[\int_t^{t+\delta} (F(s, X_s^{t,x}, Z_s^1) - F(s, x, Z_s^2)) ds | \mathcal{F}_t]| \\ & \leq CE[\int_t^{t+\delta} [\rho_0(|X_s^{t,x} - x|) + |Z_s^1 - Z_s^2|] ds | \mathcal{F}_t] \\ & \leq CE[\int_t^{t+\delta} \rho_0(|X_s^{t,x} - x|) ds | \mathcal{F}_t] + C\delta^{\frac{1}{2}} E[\int_t^{t+\delta} |Z_s^1 - Z_s^2|^2 ds | \mathcal{F}_t]^{\frac{1}{2}} \\ & \leq C\delta^{\frac{3}{2}}. \end{aligned}$$

Thus, the proof is complete. \square

Now we are able to give the proof of Theorem 6.1:

Proof. Obviously, $u(T, x) = E[\Phi(X_T^{0,x_0}, x)]$, $x \in \mathbb{R}^n$. Let us show that u is a viscosity supersolution (respectively, subsolution). For this we suppose that $\varphi \in C_{l,b}^3([0, T] \times \mathbb{R}^n)$ and $(t, x) \in [0, T) \times \mathbb{R}^n$ are such that $u - \varphi$ attains its minimum (respectively, maximum) at (t, x) . Notice that we can replace the condition of a local minimum (respectively, maximum) by that of a global one in the definition of the viscosity supersolution (respectively, subsolution) since u is continuous and of at most linear growth. Moreover, without loss of generality we may also suppose that $\varphi(t, x) = u(t, x)$. Then, due to the DPP (see (5.10)),

$$\varphi(t, x) = u(t, x) = G_{t,t+\delta}^{t,x}[u(t+\delta, X_{t+\delta}^{t,x})], \quad 0 \leq \delta \leq T - t,$$

and from $u \geq \varphi$ (respectively, $u \leq \varphi$) and the monotonicity property of $G_{t,t+\delta}^{t,x}[\cdot]$ (see Lemma 2.2 and Theorem 3.2) we obtain

$$G_{t,t+\delta}^{t,x}[\varphi(t+\delta, X_{t+\delta}^{t,x})] - \varphi(t, x) \leq 0 \text{ (respectively, } \geq 0), \quad \text{P-a.s.}$$

Thus, from Lemma 6.1,

$$Y_t^1 \leq 0 \text{ (respectively, } \geq 0), \quad \text{P-a.s.,}$$

and furthermore, from Lemma 6.2 we have

$$\int_t^{t+\delta} F(s, x, 0) ds = Y_t^2 \leq C\delta^{\frac{3}{2}} \text{ (respectively, } \geq -C\delta^{\frac{3}{2}}), \quad \text{P-a.s.}$$

It then follows that

$$F(t, x, 0) \leq 0 \text{ (respectively, } \geq 0)$$

and from the definition of F we see that u is a viscosity supersolution (respectively, subsolution) of equation (6.1). Finally, we prove that u is a viscosity solution of equation (6.1). \square

7 Viscosity Solution of PDE: Uniqueness Theorem

The objective of this section is to study the uniqueness of the viscosity solution of PDE (6.1),

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) + Au(t, x) + E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), Du(t, x).E[\sigma(t, X_t^{0,x_0}, x)])] = 0, \\ \quad (t, x) \in [0, T) \times \mathbb{R}^n, \\ u(T, x) = E[\Phi(X_T^{0,x_0}, x)], \quad x \in \mathbb{R}^n, \end{cases} \quad (7.1)$$

with

$$Au(t, x) := \frac{1}{2} \text{tr}(E[\sigma(t, X_t^{0,x_0}, x)]E[\sigma(t, X_t^{0,x_0}, x)]^T D^2 u(t, x)) + Du(t, x).E[b(t, X_t^{0,x_0}, x)].$$

Here the functions b, σ, f and Φ are supposed to satisfy (H4.1), (H5.1) and (H5.2), respectively.

We will prove the uniqueness for equation (7.1) in the space $C_p([0, T] \times \mathbb{R}^n)$ of continuous functions with at most polynomial growth. In an earlier work Barles, Buckdahn and Pardoux [3] introduced the space of continuous functions

$\Theta = \{\varphi \in C([0, T] \times \mathbb{R}^n) : \exists \tilde{A} > 0 \text{ such that } \lim_{|x| \rightarrow \infty} |\varphi(t, x)| \exp\{-\tilde{A}[\log((|x|^2 + 1)^{\frac{1}{2}})]^2\} = 0, \text{ uniformly in } t \in [0, T]\}.$

Its growth condition is slightly weaker than the assumption of polynomial growth but more restrictive than that of exponential growth. They proved in Θ the uniqueness of the viscosity solution of an integro-partial differential equation associated with a decoupled FBSDE with jumps. It was shown in [3] that this kind of growth condition is optimal for the uniqueness. However, as the following example shows, we cannot hope to have this class Θ for the uniqueness also for our type of PDE.

Example 7.1 Let $n = d = 1$, $\sigma(s, x', x) = \sigma x$, $b(s, x', x) = \frac{\sigma^2}{2}x$ and $x_0 = 1$. Then $X_s^{0, x_0} = \exp\{\sigma B_s\}$, $s \in [0, T]$, and for $f(s, x', x, y', y, z) = y'$, PDE (7.1) takes the form

$$\begin{cases} \frac{\partial}{\partial t}u(t, x) + \frac{\sigma^2}{2}x \frac{\partial}{\partial x}u(t, x) + \frac{\sigma^2}{2}x^2 \frac{\partial^2}{\partial x^2}u(t, x) + E[u(t, \exp\{\sigma B_t\})] = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\ u(T, x) = E[\Phi(\exp\{\sigma B_T\}, x)], & x \in \mathbb{R}. \end{cases}$$

But, for the growth function

$$\tilde{\chi}(t, x) = \exp\{\tilde{A}[\log((|x|^2 + 1)^{\frac{1}{2}})]^2\}, \quad (t, x) \in [0, T) \times \mathbb{R},$$

which comes from the definition of Θ , we have

$$\begin{aligned} E[\tilde{\chi}(t, X_t^{0, x_0})] &= E[\exp\{\tilde{A}[\log((|\exp\{\sigma B_t\}|^2 + 1)^{\frac{1}{2}})]^2\}] \\ &\geq E[\exp\{\tilde{A}\sigma^2 B_t^2\}] \\ &= +\infty, \quad \text{if } t \in [\frac{1}{2A\sigma^2}, T]. \end{aligned}$$

As the example shows, for a function $u \in \Theta$, the coefficient $E[f(t, X_t^{0, x_0}, x, u(t, X_t^{0, x_0}), y, z)]$ may be not well defined. This is the reason why we restrict the study of the uniqueness to the smaller class $C_p([0, T] \times \mathbb{R}^n)$.

Theorem 7.1. *We assume that (H4.1), (H5.1) and (H5.2) hold. Let u_1 (resp., u_2) $\in C_p([0, T] \times \mathbb{R}^n)$ be a viscosity subsolution (resp., supersolution) of equation (7.1). Then we have*

$$u_1(t, x) \leq u_2(t, x), \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^n.$$

The proof of the theorem will be prepared by the following auxiliary lemmata.

Lemma 7.1. *Let K be a Lipschitz constant of $f(t, x', \cdot, \cdot, \cdot, \cdot)$, uniformly in (t, x') from (H5.1)-(ii), and let $\nu > K$. Then, if $u \in C_p([0, T] \times \mathbb{R}^n)$ is a viscosity subsolution (respectively, supersolution) of PDE (7.1) the function $\bar{u}(t, x) = u(t, x)e^{\nu t}$, $(t, x) \in [0, T] \times \mathbb{R}^n$, is a viscosity subsolution (respectively, supersolution) of the following PDE:*

$$\begin{cases} \frac{\partial}{\partial t}\bar{u}(t, x) + D\bar{u}(t, x) \cdot E[b(t, X_t^{0, x_0}, x)] + \frac{1}{2}\text{tr}(E[\sigma(t, X_t^{0, x_0}, x)]E[\sigma(t, X_t^{0, x_0}, x)]^T D^2\bar{u}(t, x)) \\ + E[\bar{f}(t, X_t^{0, x_0}, x, \bar{u}(t, X_t^{0, x_0}), \bar{u}(t, x), D\bar{u}(t, x) \cdot E[\sigma(t, X_t^{0, x_0}, x)])] = 0, & (t, x) \in [0, T) \times \mathbb{R}^n; \\ \bar{u}(T, x) = E[\Phi(X_T^{0, x_0}, x)]e^{\nu T}, & x \in \mathbb{R}^n, \end{cases} \quad (7.2)$$

where $\bar{f}(t, x', x, y', y, z) = e^{\nu t}f(t, x', x, e^{-\nu t}y', e^{-\nu t}y, e^{-\nu t}z) - \nu y$, $(t, x', x, y', y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, conserves the properties (H5.1) and (H5.2) of f and is, moreover, strictly decreasing in y :

$$\begin{aligned} \bar{f}(t, x', x, y', y_1, z) - \bar{f}(t, x', x, y', y_2, z) &\leq -(\nu - K)(y_1 - y_2), \\ &\text{for all } (t, x', x, y', z) \text{ and all } y_1, y_2 \in \mathbb{R} \text{ with } y_1 \geq y_2. \end{aligned}$$

The proof of this well-known transformation is straight-forward and is hence omitted.

Lemma 7.2. *Let $u_1 \in C_p([0, T] \times \mathbb{R}^n)$ be a viscosity subsolution and $u_2 \in C_p([0, T] \times \mathbb{R}^n)$ be a viscosity supersolution of equation (7.2). Then the function $\omega := u_1 - u_2 \in C_p([0, T] \times \mathbb{R}^n)$ is a viscosity subsolution of the equation*

$$\begin{cases} -\nu\omega(t, x) + \frac{\partial}{\partial t}\omega(t, x) + \frac{1}{2}\text{tr}(\tilde{\sigma}\tilde{\sigma}^T(t, x)D^2\omega) + D\omega \cdot \tilde{b}(t, x) + K|\omega(t, x)| + \\ \quad + KE[(\omega(t, X_t^{0, x_0}))^+] + K|D\omega \cdot \tilde{\sigma}(t, x)| = 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ \omega(T, x) = 0, & x \in \mathbb{R}^n. \end{cases} \quad (7.3)$$

(See (6.2) for the notations \tilde{b} and $\tilde{\sigma}$.)

The proof of this lemma follows from that of Lemma 3.7 in [3], it turns out to be even simpler because contrary to Lemma 3.7 in [3] we don't have any integral part generated by a diffusion process with jumps here in the equations (5.1) and (5.2).

Lemma 7.3. *For any $A > 1$, there exists $C_1 > 0$ such that the function*

$$\chi(t, x) = Ae^{C_1(T-t)}\psi(x),$$

with

$$\psi(x) = (|x|^2 + 1)^{\frac{p}{2}}, \quad x \in \mathbb{R}^n, \quad p > 1,$$

satisfies

$$\begin{aligned} & \frac{\partial}{\partial t}\chi(t, x) + \frac{1}{2}\text{tr}(\tilde{\sigma}\tilde{\sigma}^T(t, x)D^2\chi) + D\chi \cdot \tilde{b}(t, x) + K\chi(t, x) + \\ & K|D\chi(t, x) \cdot \tilde{\sigma}(t, x)| + KE[\chi(t, X_t^{0, x_0})] < 0 \quad \text{in } [0, T] \times \mathbb{R}^n. \end{aligned} \quad (7.4)$$

Proof. By direct computation we first deduce the following estimates for the first and second derivatives of ψ :

$$|D\psi(x)| = p \frac{\psi(x)}{|x|^2 + 1}|x|, \quad |D^2\psi(x)| \leq p^2 \frac{\psi(x)}{|x|^2 + 1}, \quad x \in \mathbb{R}^n,$$

and

$$\frac{\partial}{\partial t}\chi(t, x) = -C_1\chi(t, x), \quad E[\psi(X_t^{0, x_0})] \leq C_p\psi(x), \quad x \in \mathbb{R}^n.$$

The latter estimate is a direct consequence of (4.2). Taking into account that the coefficients b, σ are of most linear growth, these estimates imply for all $t \in [0, T]$,

$$\begin{aligned} & \frac{\partial}{\partial t}\chi(t, x) + \frac{1}{2}\text{tr}(\tilde{\sigma}\tilde{\sigma}^T(t, x)D^2\chi) + D\chi \cdot \tilde{b}(t, x) + K\chi(t, x) + \\ & K|D\chi(t, x) \cdot \tilde{\sigma}(t, x)| + KE[\chi(t, X_t^{0, x_0})] \\ & \leq -\chi(t, x)\{C_1 - p^2C - K - KC_p\} \\ & = -\chi(t, x) < 0, \quad \text{if } C_1 := p^2C + K + KC_p + 1 \text{ large enough.} \end{aligned}$$

□

Now we can prove the uniqueness theorem.

Proof of Theorem 7.1. From Lemma 7.1 we only need to prove that: If \bar{u}_1 (respectively, \bar{u}_2) $\in C_p([0, T] \times \mathbb{R}^n)$ is a viscosity subsolution (respectively, supersolution) of equation (7.2) then we have

$$\bar{u}_1(t, x) \leq \bar{u}_2(t, x), \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^n.$$

Let us put $\omega := \bar{u}_1 - \bar{u}_2$. Since $\omega \in C_p([0, T] \times \mathbb{R}^n)$, there exist some constants $C > 0$ and $q \geq 1$ such that $|\omega(t, x)| \leq C(1 + |x|)^q$, $(t, x) \in [0, T] \times \mathbb{R}^n$. In the definition of

$$\chi(t, x) = Ae^{C_1(T-t)}(|x|^2 + 1)^{\frac{p}{2}}$$

we now choose $p > q$. Then, for any $\alpha > 0$, $\omega(t, x) - \alpha\chi(t, x)$ is bounded from above in $[0, T] \times \mathbb{R}^n$ and

$$M := \max_{[0, T] \times \mathbb{R}^n} (\omega - \alpha\chi)(t, x)e^{-K(T-t)}$$

is achieved at some point $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ (depending on α). We now have to distinguish between two cases:

For the first case we suppose that $\omega(t_0, x_0) \leq 0$, for all $\alpha > 0$.

Then, obviously $M \leq 0$ and $\bar{u}_1(t, x) - \bar{u}_2(t, x) \leq \alpha\chi(t, x)$ in $[0, T] \times \mathbb{R}^n$. Consequently, letting α tend to zero we obtain

$$\bar{u}_1(t, x) \leq \bar{u}_2(t, x), \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^n.$$

For the second case we assume that there exists some $\alpha > 0$ such that $\omega(t_0, x_0) > 0$.

We notice that $\omega(t, x) - \alpha\chi(t, x) \leq (\omega(t_0, x_0) - \alpha\chi(t_0, x_0))e^{-K(t-t_0)}$ in $[0, T] \times \mathbb{R}^n$. Then, putting

$$\varphi(t, x) = \alpha\chi(t, x) + (\omega - \alpha\chi)(t_0, x_0)e^{-K(t-t_0)}, \quad (t, x) \in [0, T] \times \mathbb{R}^n,$$

we get $\omega - \varphi \leq 0 = (\omega - \varphi)(t_0, x_0)$ in $[0, T] \times \mathbb{R}^n$. Consequently, since ω is a viscosity subsolution of (7.3) from Lemma 7.2,

$$\begin{aligned} & -\nu\omega(t_0, x_0) + \frac{\partial}{\partial t}\varphi(t_0, x_0) + \frac{1}{2}\text{tr}(\tilde{\sigma}\tilde{\sigma}^T(t_0, x_0)D^2\varphi(t_0, x_0)) + D\varphi(t_0, x_0).\tilde{b}(t_0, x_0) + \\ & K|\varphi(t_0, x_0)| + K|D\varphi(t_0, x_0).\tilde{\sigma}(t_0, x_0)| + KE[(\varphi(t_0, X_{t_0}^{0, x_0}))^+] \geq 0. \end{aligned}$$

Moreover, due to our assumption that $\omega(t_0, x_0) > 0$ and since $\omega(t_0, x_0) = \varphi(t_0, x_0)$ we can replace $K|\varphi(t_0, x_0)|$ by $K\varphi(t_0, x_0)$ in the above formula. Then, from the definition of φ and Lemma 7.3,

$$\begin{aligned} 0 & \leq -\nu\omega(t_0, x_0) + \frac{\partial}{\partial t}\varphi(t_0, x_0) + \frac{1}{2}\text{tr}(\tilde{\sigma}\tilde{\sigma}^T(t_0, x_0)D^2\varphi(t_0, x_0)) + D\varphi(t_0, x_0).\tilde{b}(t_0, x_0) + \\ & K|\varphi(t_0, x_0)| + K|D\varphi(t_0, x_0).\tilde{\sigma}(t_0, x_0)| + KE[(\varphi(t_0, X_{t_0}^{0, x_0}))^+] \\ & = \alpha\frac{\partial}{\partial t}\chi(t_0, x_0) + \alpha K\chi(t_0, x_0) + \alpha\frac{1}{2}\text{tr}(\tilde{\sigma}\tilde{\sigma}^T(t_0, x_0)D^2\chi(t_0, x_0)) + \alpha D\chi(t_0, x_0).\tilde{b}(t_0, x_0) + \\ & \alpha K|D\chi(t_0, x_0).\tilde{\sigma}(t_0, x_0)| + KE[(\alpha\chi(t_0, X_{t_0}^{0, x_0}) + (\omega - \alpha\chi)(t_0, x_0))^+] - \nu\omega(t_0, x_0) \\ & \leq \alpha\left\{\frac{\partial\chi}{\partial t}(t_0, x_0) + K\chi(t_0, x_0) + \frac{1}{2}\text{tr}(\tilde{\sigma}\tilde{\sigma}^T(t_0, x_0)D^2\chi(t_0, x_0)) + D\chi(t_0, x_0).\tilde{b}(t_0, x_0) \right. \\ & \quad \left. + K|D\chi(t_0, x_0).\tilde{\sigma}(t_0, x_0)| + KE[\chi(t_0, X_{t_0}^{0, x_0})]\right\} < 0, \end{aligned}$$

since $\nu > K$, which is a contradiction. Thus, the proof is complete. \square

Remark 7.1. Obviously, since the value function $u(t, x)$ is of at most linear growth it belongs to $C_p([0, T] \times \mathbb{R}^n)$, and so $u(t, x)$ is the unique viscosity solution in $C_p([0, T] \times \mathbb{R}^n)$ of equation (6.1).

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